Fall semester 2012
September 28, 2012

MATHEMATICS DEPARTMENT CONTEST

The Department of Mathematics invites all CUA students to compete, for the fun of it, in a mathematics contest. The contest consists of mathematical problems or puzzles which can be understood by anyone with the usual high school mathematics background. The most successful contestants will be invited to the Mathematics Department end-of-semester party to receive prizes. There will be prizes for the students who solve the most problems and for those who submit the most interesting or original solutions (even if for only one problem). Send your solutions by November 19, 2012 to Dr. Alexander Levin at the Mathematics Department in McMahon Hall, room 207. They need not be typed but should be legible and should show or explain how you solved the puzzle.

Problem 1. What is greater, the number of seven-digit numbers whose decimal representations contain 1 or the number of seven-digit numbers whose decimal representations do not contain 1? Justify your answer.

Problem 2. Prove that if x and y are two real numbers such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$, then $\dfrac{x}{1+y} + \dfrac{y}{1+x} \leq 1$.

Problem 3. A square is cut into several equal right triangles with legs of length $3\,\text{in}$ and $4\,\text{in}$ and several equal $2\,\text{in}\times 2\,\text{in}$ squares. Prove that the number of the right triangles is even.

Problem 4. There are 2012 inhabitants in a small town. Five of them are mathematicians but just a few people in the town know who are the mathematicians. Mr. Johns, who is visiting the town, has asked every inhabitant to name four people he or she thinks are mathematicians. Of course, Mr. Johns knows that every mathematician has named the four other mathematicians. Prove that based on these data, Mr. Johns can choose an inhabitant who is not a mathematician.

Problem 5. Two students are playing the following game. They put quarters on a rectangular table taking turns. If a player cannot put the quarter (because there is no space on the table), he loses the game. Who is going to win, the student who places the first coin or his opponent? What is the winning strategy? (One cannot put a quarter on the top of another coin or move the coins that are already on the table.)

Problem 6. Consider a sequence of numbers $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$. Is it possible to choose seventeen distinct numbers from this sequence such that the sum of the numbers is equal to 1? Can all such seventeen numbers have odd denominators?