

# MATHEMATICS DEPARTMENT CONTEST

Fall semester 2018

The Department of Mathematics invites all CUA students to compete, for the fun of it, in a mathematics contest. The contest consists of mathematical problems or puzzles which can be understood by anyone with the usual high school mathematics background. The most successful contestants will be invited to the Mathematics Department end-of-semester party to receive prizes. There will be prizes for the students who solve the most problems and for those who submit the most interesting or original solutions (even if for only one problem).

Send your solutions by **November 27, 2018** to Dr. Alexander Levin at the Mathematics Department in Aquinas Hall, room 116. They need not be typed but should be legible and should show or explain how you solved the puzzle.

**Problem 1.** Prove that the polynomial  $f(x) = (x - a)^2(x - b)^2 + 1$ , where  $a$  and  $b$  are integers, cannot be factored into a product of two polynomials with integer coefficients.

**Problem 2.** Prove that the area of a triangle whose two sides have lengths  $a$  and  $b$  cannot exceed  $\frac{a^2 - ab + b^2}{2}$ .

**Problem 3.** There are 2018 points in the plane such that for every three points, there is a circle of radius 1 containing these points. Prove that there is a circle of radius 1 that contains all 2018 points.

**Problem 4.** Let  $X$  be a 120-digit positive integer. Suppose we permute its first twelve digits in all possible ways and then randomly choose 120 numbers from the set of all 120-digit positive integers obtained by such permutations. Prove that the sum of the chosen numbers is divisible by 120.

**Problem 5.** Twenty-five people live on some island: knights who always tell the truth, liars who always lie, and knaves who alternate between truths and lies. (If a knave tells the truth answering a question, he (or she) lies answering the next question and then again tells the truth answering the following question, etc.) Each person on the island was asked three questions: "Are you a knight?" (the first question), "Are you a knave?" (the second question), and "Are you a liar?" (the third question). Twenty-one people answered "Yes" to the first question, seventeen people answered "Yes" to the second question, and nine people answered "Yes" to the third question. How many knaves live on the island? Justify your answer.

**Problem 6.** Prove that in any set of ten integers, one can choose several numbers whose sum is divisible by 10.