Spring semester 2019
January 16, 2019

MATHEMATICS DEPARTMENT CONTEST

The Department of Mathematics invites all CUA students to compete, for the fun of it, in a mathematics contest. The contest consists of mathematical problems or puzzles which can be understood by anyone with the usual high school mathematics background. The most successful contestants will be invited to the Mathematics Department end-of-semester party to receive prizes. There will be prizes for the students who solve the most problems and for those who submit the most interesting or original solutions (even if for only one problem).

Send your solutions by **April 23, 2019** to Dr. Alexander Levin at the Mathematics Department in Aquinas Hall, room 116. They need not be typed but should be legible and should show or explain how you solved the puzzle.

Problem 1. Every Halloween, the Winchester family hosts their annual Halloween Pumpkin Pie Bake Off. However, this year a rotten thief has stolen and hidden away the prized pumpkin pie! Detectives come and narrow it down to three suspects: John, Sam, and Chris. Under questioning, each suspect makes two statements.

John: (1) It was not me. (2) It was Chris.

Sam: (1) It was not Chris. (2) It was John.

Chris: (1) It was not me. (2) It was not Sam.

It turned out that both statements of one of the suspects are true, both statements of another suspect are false, and the third suspect made one true and one false statement. Who is the pie thief?

Problem 2. Consider a circle of radius 1 in. Is it possible to find seven points in this circle such that the distance between any two different points is greater than 1 in? Justify your answer.

Problem 3. A square of an integer ends with four equal digits. What are these digits? Justify your answer.

Problem 4. An equilateral triangle ABC is inscribed into a circle. (The vertices A, B, and C lie on the circumference of the circle.) The arcs AB and BC are divided into two equal parts (arcs) by points D and E, respectively. (D lies on the circumference in the middle of the arc AB, E lies on the circumference in the middle of the arc BC.) Prove that the sides of the triangle divide the segment DE into three equal parts.

Problem 5. Consider ten real numbers $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4$, and $b5$ such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 1$ and $b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 = 1$. Prove that

$$-1 \leq a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 \leq 1.$$

Problem 6. Prove that if n and k are positive integers and k is odd, then the number $a = 1^k + 2^k + 3^k + \cdots + n^k$ is divisible by the number $b = 1 + 2 + 3 + \cdots + n$.