SAMPLE PROBLEMS FOR THE TAKE-HOME COMPONENT OF THE COMPREHENSIVE EXAM

Problem 1. Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \quad (n = 1, 2, 3, \dots)$. Prove that the sequence $\{x_n\}$ converges.
[*Hint*: Show, first, that $x > \ln(1 + x)$ for any $x > 0$. Then show that the sequence $\{x_n\}$ increases and $x_{n+1} - x_n < \frac{1}{2(n+1)^2}$. Use the theorem about the convergence and divergence of p-series to complete the proof.]

Problem 2. (2 points). Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be absolutely convergent series of real numbers. Prove that the series $\sum_{n=1}^{\infty} \sqrt{|a_n b_n|}$ converges.

Problem 3. Prove that if a function $f(x)$ is continuous on a segment $[a, b]$ and $\int_a^b f(x)dx = 0$, then there exists a point $c \in (a, b)$ such that

$$\int_a^c f(t)dt = f(c).$$

[*Hint*: Apply Rolle's Theorem to the function $g(x) = e^{-x} \int_a^x f(t)dt$ $(a \le x \le b)$.]

Problem 4. Prove that the function $f(x) = \log_x(x + 1)$ decreases on the interval $(1, +\infty)$.

Problem 5. Let functions $f(x)$ and $g(x)$ be continuous on an interval $[a, b]$. Prove that if $\int_a^b f^2(x)dx = 0$, then $\int_a^b f(x)g(x)dx = 0$.
[*Hint*: Show, first, that $\int_a^b (tf(x) + g(x))^2 dx \ge 0$ for any real number t. Then show that $2\left|\int_a^b f(x)g(x)dx\right| \le t\int_a^b f^2(x)dx + \frac{1}{t}\int_a^b g^2(x)dx$ for any $t > 0$.]

Problem 6. Determine whether the series

$$\sum_{n=2}^{\infty} (-1)^n \int_n^{n+1} \frac{dx}{\ln^2 x}$$

is absolutely convergent, conditionally convergent, or divergent.

Problem 7. Let $f(x,y) = \sqrt[3]{xy}$ be a function of two independent real variables x and y. Find all directions $\mathbf{u}$ in which the directional derivative $D_{\mathbf{u}}f(0,0)$ of the function $f(x,y)$ at the point $(0,0)$ exists.

Problem 8. Find all 2×2-matrices A with real entries such that $A = A^{-1}$.

Problem 9. Let G be a group with identity e. Prove that if G has less than five subgroups (including the trivial subgroups G and $\{e\}$), then the group G is cyclic.

Problem 10. Let $\mathbb{P}_n$ (n is a positive integer) be the vector space of all polynomials with real coefficients whose degree is less than n ($\mathbb{P}_n$ is considered as a vector space over the field of real numbers $\mathbb{R}$). Let $V = \{f(x) \in \mathbb{P}_n \,|\, f(1) = 0\}$. Check that V is a vector subspace of $\mathbb{P}_n$ and find a vector subspace W of $\mathbb{P}_n$ such that $\mathbb{P}_n = V \bigoplus W$ (that is, $\mathbb{P}_n$ is the direct sum of V and W).

Problem 11. Suppose that the order of some finite Abelian group G is divisible by 42. Prove that G has a cyclic subgroup of order 42.

Problem 12. Consider the linear transformation

$$L(\mathbf{x}) = \det\,(\mathbf{x}, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n)$$

from $\mathbf{R}^n$ to $\mathbf{R}$, where $\mathbf{v}_2$, $\mathbf{v}_3$, ... ,$\mathbf{v}_n$ are linearly independent vectors in $\mathbf{R}^n$. Describe the range and the kernel of this linear transformation, and determine their dimensions.