SAMPLE PROBLEMS FOR THE TAKE-HOME COMPONENT OF THE COMPREHENSIVE EXAM

**Problem 1.** Let \( x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \) \((n = 1, 2, 3, \ldots)\). Prove that the sequence \( \{x_n\} \) converges.

*Hint:* Show, first, that \( x > \ln(1 + x) \) for any \( x > 0 \). Then show that the sequence \( \{x_n\} \) increases and \( x_{n+1} - x_n < \frac{1}{2(n+1)^2} \). Use the theorem about the convergence and divergence of \( p \)-series to complete the proof.

**Problem 2.** (2 points). Let \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) be absolutely convergent series of real numbers. Prove that the series \( \sum_{n=1}^{\infty} \sqrt{|a_n b_n|} \) converges.

**Problem 3.** Prove that if a function \( f(x) \) is continuous on a segment \([a, b]\) and \( \int_a^b f(x)dx = 0 \), then there exists a point \( c \in (a, b) \) such that \( \int_a^c f(t)dt = f(c) \).

*Hint:* Apply Rolle’s Theorem to the function \( g(x) = e^{-x} \int_a^x f(t)dt \) \((a \leq x \leq b)\).

**Problem 4.** Prove that the function \( f(x) = \log_x (x + 1) \) decreases on the interval \((1, +\infty)\).

**Problem 5.** Let functions \( f(x) \) and \( g(x) \) be continuous on an interval \([a, b]\). Prove that if \( \int_a^b f^2(x)dx = 0 \), then \( \int_a^b f(x)g(x)dx = 0 \).

*Hint:* Show, first, that \( \int_a^b (tf(x) + g(x))^2 dx \geq 0 \) for any real number \( t \). Then show that \( 2 \left| \int_a^b f(x)g(x)dx \right| \leq t \int_a^b f^2(x)dx + \frac{1}{t} \int_a^b g^2(x)dx \) for any \( t > 0 \).

**Problem 6.** Determine whether the series \( \sum_{n=2}^{\infty} (-1)^n \int_n^{n+1} \frac{dx}{\ln^2 x} \)
is absolutely convergent, conditionally convergent, or divergent.

Problem 7. Let \( f(x, y) = \sqrt[3]{xy} \) be a function of two independent real variables \( x \) and \( y \). Find all directions \( u \) in which the directional derivative \( D_u f(0, 0) \) of the function \( f(x, y) \) at the point \((0, 0)\) exists.

Problem 8. Find all \( 2 \times 2 \)-matrices \( A \) with real entries such that \( A = A^{-1} \).

Problem 9. Let \( G \) be a group with identity \( e \). Prove that if \( G \) has less than five subgroups (including the trivial subgroups \( G \) and \( \{e\} \)), then the group \( G \) is cyclic.

Problem 10. Let \( \mathbb{P}_n \) \((n \) is a positive integer\) be the vector space of all polynomials with real coefficients whose degree is less than \( n \) \((\mathbb{P}_n \) is considered as a vector space over the field of real numbers \( \mathbb{R} \)). Let \( V = \{ f(x) \in \mathbb{P}_n \mid f(1) = 0 \} \). Check that \( V \) is a vector subspace of \( \mathbb{P}_n \) and find a vector subspace \( W \) of \( \mathbb{P}_n \) such that \( \mathbb{P}_n = V \oplus W \) \( \) (that is, \( \mathbb{P}_n \) is the direct sum of \( V \) and \( W \)).

Problem 11. Suppose that the order of some finite Abelian group \( G \) is divisible by 42. Prove that \( G \) has a cyclic subgroup of order 42.

Problem 12. Consider the linear transformation

\[
L(x) = \det (x, v_2, v_3, ..., v_n)
\]

from \( \mathbb{R}^n \) to \( \mathbb{R} \), where \( v_2, v_3, ..., v_n \) are linearly independent vectors in \( \mathbb{R}^n \). Describe the range and the kernel of this linear transformation, and determine their dimensions.