Mathematics Department Contest
Spring, 2008

The Department of Mathematics invites all CUA students to compete, for the fun of it, in a mathematics contest. The contest consists of mathematical problems or puzzles which can be understood by anyone with the usual high school mathematics background. The most successful contestants will be invited to the Mathematics Department end-of-semester party to receive prizes. There will be prizes for the students who solve the most problems and for those who submit the most interesting or original solutions (even if for only one problem).

Send your solutions by April 23, 2008 to Dr. Alexander Levin at the Mathematics Department in room 207 MCM. They need not be typed but should be legible and should show or explain how you solved the puzzle.

Problem 1. Is it possible to find integers x and y such that $x^3 + 21y^2 + 5 = 0$? Justify your answer.

Problem 2. A 50 inch $\times$ 50 inch square is divided by horizontal and vertical lines into 2500 1 inch $\times 1$ inch square cells. Each cell is painted in one of four available colors. Prove that there is a cell A such that one can find a cell above and a cell below A in the same column as A which have the same color as A, and that one can find a cell to the left of A and to the right of A in the same row as A which also have the same color as A.

Problem 3. What is the smallest positive integer N such that if one writes any N different two-digit numbers, there will be a number with equal digits?

Problem 4. A flea jumps in the plane in such a way that the distance covered by its second jump is two times shorter than the distance covered by the first jump, the distance covered by its third jump is two times shorter than the distance covered by the second jump, etc. Can the flea visit the same point of the plane twice? The flea is allowed to jump any finite number of times. Justify your answer.

Problem 5. Let us consider three circles C_1, C_2 and C_3 whose centers lie on the same line. Suppose that no two of the circles intersect each other. There is a fourth circle D whose circumference touches the circumferences of the each of the three circles C_1, C_2 and C_3. Prove that the radius of D is greater than one of the radii of the circles C_1, C_2 or C_3.

Problem 6. A positive integer x is written on the blackboard. It is allowed to replace this number by either $2x + 4$, $3x + 8$ or $x^2 + 5x$. Is it possible to obtain 2008 after several replacements of this kind? Justify your answer.