MATHEMATICS COMPREHENSIVE EXAM: IN-CLASS COMPONENT

The following is the list of questions for the oral exam. At the same time, these questions represent all topics for the written exam.

The procedure for the oral exam is as follows. A student receives two questions from the list below and a problem. The questions and the problem will belong to two of the three different sections (Calculus and Introductory Analysis, Linear Algebra, and Abstract Algebra). The problem will belong to the course of the students choice. The student will have one hour for preparing the answers.

This is an OPEN BOOK EXAM and a student can use any literature in his (her) preparation. Then the student answers the questions and presents the solution of the problem at the blackboard. (The student can write his or her notes on the blackboard during the preparation.) The examiners will ask questions during the presentation; these questions will be related to the presentation, but there might be some additional questions on the topics outlined by the list of questions below.

The students who take written exam will receive a test with eight problems: two problems in Calculus and Introductory Analysis, two problems in Linear Algebra, two problems in Abstract Algebra, and two problems in the area (course) of the students choice. The student must complete the exam within 3 hours. This is also an OPEN BOOK EXAM and a student can use any literature at the test.

I. **Calculus and Introductory Analysis**

1. The definition of the limit of a function of one variable. Basic properties of limits. Examples.
2. Continuity of functions of one variable. Basic properties of continuous functions. Intermediate Value Theorem.
3. Uniform continuity of functions of one variable. An example of a function which is continuous but not uniformly continuous on an interval.
4. The definition of the derivative of a function of one variable. Basic rules of differentiation. The relationship between the continuity and differentiability.
5. The Mean Value Theorem for derivatives. Monotonicity Theorem.
6. Chain rule and implicit differentiation of functions of one variable. Logarithmic differentiation.

7. Cauchys Mean Value Theorem. LHpitals Rule.
8. Local extrema of functions of one variable. Necessary and sufficient conditions for local extrema (first and second derivative tests).
9. The definition and basic properties of the Riemann integral. Mean Value Theorem for integrals.
10. The Fundamental Theorem of Calculus.
11. Geometric applications of the integral (the area of a plane region, the volume of a solid of revolution, and the length of a plane curve).
12. Improper Riemann integration (integrals with infinite limits of integration and integrals with unbounded integrands).
13. The definition and basic properties of the limit of a sequence. The theorem on the limit of a monotonic bounded sequence.
14. Infinite series of real numbers. n-th Term test for divergence. Linearity of convergent series. Geometric series.
15. Cauchy sequences. Cauchy criterion for infinite series.
16. Integral, Comparison, and Ratio tests for series with nonnegative terms. Convergent and divergent p-series.
17. Alternative series test. Absolutely and conditionally convergent series. Absolute Convergence Test and Absolute Ratio Test.
18. Uniform convergence of sequences of functions. The theorem on the continuity of the limit of a uniformly convergent sequence of functions.
19. Uniform convergence of series of functions. The Weierstrass M-test. The theorem on continuity of the sum of a uniformly convergent series of functions.
20. Theorems on term-by-term integration and differentiation of uniformly convergent series of functions.
21. Power series. Abels Theorem. Radius and interval of convergence of a power series. Continuity of the sum of a power series on its interval of convergence.
22. Theorems on term-by-term integration and differentiation of power series.
23. Taylor and Maclaurin series. The theorem on the uniqueness of the expansion of a function into a power series. Maclaurin series of basic elementary functions.
24. Taylors formula with remainder. Taylors theorem (the sufficient condition of the representation of a function by its Taylor series).

25. Parametric and polar equations of curves in the plane. Examples of curves given by parametric and polar equations. Area in polar coordinates.

26. Cartesian coordinates in three-dimensional space. Cylindrical and Spherical coordinates. Vectors. Basic operations on vectors. Coordinates of vectors.

27. The dot (scalar) product of vectors in three-dimensional space. Basic properties of the dot product and its coordinate expression. The cross (vector) product. Basic properties of the cross product and its coordinate expression.

28. Equations of planes and lines in three-dimensional space.

29. Vector-valued functions of a real argument and curvilinear motion. Limits, derivatives, and integrals of vector-valued functions of a real argument. Interpretation of a space curve as a trajectory of a moving point. Velocity and acceleration vectors. Curvature.

30. Limit and continuity of functions of two or more variables. Partial derivatives.

31. Surfaces in three-dimensional space. Examples. Differentiability of a function of two or more variables and the equation of the tangent plane to a surface.

32. Directional derivatives of a function of two or three variables. Basic properties of the gradient. Chain rule for functions of several variables.

33. Extrema of functions of two variables. The Critical Point Theorem. The Second Partials Test.

34. The definition and basic properties of double integral. Computations of double integrals with the use of iterated integrals (state the corresponding theorem without a proof and give an example). Double integrals in polar coordinates.

35. The definition and basic properties of line integrals. Greens Theorem in the plane. Independence of Path Theorem.

36. Scalar and vector fields. Conservative vector fields. Divergence and curl of a vector field. Independence of Path Theorem for line integrals. Finding a scalar potential function for a conservative vector field.

37. Surface integrals. The Gausss Divergence Theorem and the Stokes Theorem.

II. **Linear Algebra**

1. Systems of linear equations and their matrices. Row echelon form of a matrix. The method of Gaussian elimination.
2. Basic operations on matrices. Computation of the inverse matrix of a non-singular square $n \times n$ matrix.
3. Elementary matrices. Row equivalent matrices. Equivalent conditions for nonsingularity.
4. The determinant of a square matrix. Basic properties of determinants.
5. The adjoint of a matrix. Cramers Rule.
6. The definition of a vector space. Subspaces of a vector space. Examples. The nullspace of a matrix.
7. The span of a set of vectors (that is, elements of a vector space). Linear dependence and linear independence of vectors. Basic properties of linear dependence.
8. Bases of a vector space. The theorem on the equality of the numbers of elements of two finite bases of a vector space. The concept of dimension of a vector space.
9. Changing the coordinates of vectors under the change of a basis of a vector space.
10. Row space and column space of a matrix. The rank of a matrix. The Rank-Nullity Theorem. Consistency Theorem for systems of linear equations.
11. The definition of a linear transformation. Examples. The image and kernel of a linear transformation.
12. Matrix representations of linear transformations. Matrix Representation Theorem. Similarity of matrices. The relationship between the matrix representations of a linear transformation of a vector space in two different bases.
13. The scalar product in $\mathbf{R}^n$. Orthogonality. The orthogonal complement of a subspace of $\mathbf{R}^n$. Fundamental Subspaces Theorem for matrices.
14. The definition and basic properties of inner product spaces. Normed linear spaces. Examples. Orthonormal sets. Basic properties of orthogonal matrices.
15. Orthonormal sets and the method of least squares. Approximation of functions. The Gram-Schmidt orthogonalization process. Orthogonal polynomials.
16. Eigenvalues and eigenvectors. The characteristic polynomial of a matrix. Diagonalizable matrices. The criterion of diagonalizability of a matrix.

III. **Abstract Algebra**

1. The definition of a group. Elementary properties of groups (the uniqueness of the identity; the uniqueness of an inverse element; the associative law for more than three terms). Examples of groups.
2. The definition of a subgroup. Cyclic groups and their subgroups.
3. Permutation groups. Expansion of an element of S_n into a product of disjoint cycles. The order of a permutation written in this form.
4. Expansion of an element of S_n into a product of 2-cycles. Even and odd permutations.
5. Group homomorphisms and isomorphisms. Properties of homomorphisms and isomorphisms. Cayleys Theorem.
6. The group of automorphisms and the group of inner automorphisms of a group. The isomorphism between the groups $\mathbf{Z}_n$ and $\mathbf{U}(n)$.
7. Cosets and Lagranges Theorem. Theorem on groups of prime order. Fermats Little Theorem.
8. External direct product of groups. The order of an element in a direct product. Criterion for a finite direct product to be cyclic.
9. The groups of units modulo n as an external direct product.
10. Normal subgroups and factor groups. Examples. Theorem on the commutativity of a group G whose factor group by the center of G is cyclic.
11. Theorem on the isomorphism between the groups $G/Z(G)$ and $\mathrm{Inn}(G)$, where $Z(G)$ is the center of a group G. Cauchys theorem on Abelian groups.
12. Internal direct product of subgroups. The isomorphism of internal and external direct products.
13. Properties of subgroups under homomorphisms. Isomorphism theorems for groups.
14. Fundamental Theorem of finite Abelian groups. Existence of subgroups of finite Abelian groups. (Prove that if G is an Abelian group of order n and $d|n$, then G contains a subgroup of order d.) The isomorphism classes of Abelian groups.
15. The definition of a ring. Subrings and ideals of a ring. Factor rings. The definition of an integral domain. Examples.
16. Homomorphisms of rings. Theorems on ring isomorphisms.
17. Prime and maximal ideals of commutative rings.

18. Characteristic of a field. Prove that every field contains a subfield isomorphic to either the field of rational number $\mathbf{Q}$ or the finite field $\mathbf{Z}_p$ where p is a prime number.